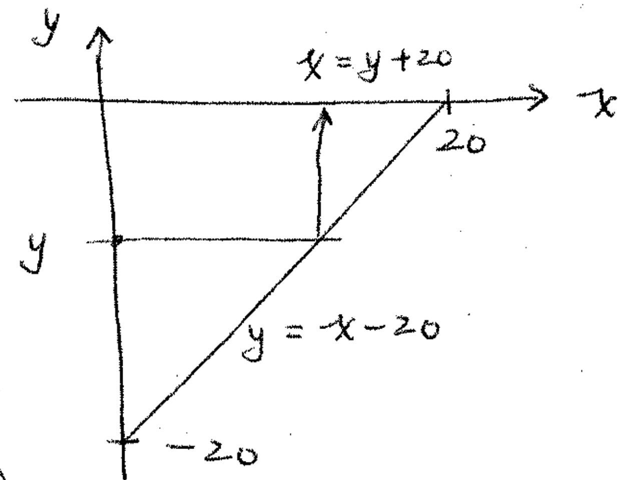
Q1

Volume over the foyer



$F: \; -20 \le y \le 0$

$\quad\; 0 \le x \le y + 20$

$$V = \iint_F f(x,y)\, dA$$

$$= \int_{-20}^{0} \int_{0}^{y+20} \left[14 - \frac{1}{100}(x^2 + y^2) \right] dx\, dy$$

$$= \int_{-20}^{0} \left[14x - \frac{1}{100}\left(\frac{x^3}{3} + xy^2 \right) \right]_0^{y+20} dy$$

$$= \int_{-20}^{0} \left[14(y+20) - \frac{1}{100}\left(\frac{(y+20)^3}{3} + (y+20)y^2 \right) \right] dy$$

$$u = y + 20 \Rightarrow du = dy$$

$$= \int_{0}^{20} \left[14u - \frac{1}{100}\left(\frac{u^3}{3} + u(u-20)^2 \right) \right] du$$

$$= \int_{0}^{20} \left[14u - \frac{1}{100}\left(\frac{u^3}{3} + u(u^2 - 40u + 400) \right) \right] du$$

$$= \int_0^{20} \left[14u - \frac{1}{100}\left(\frac{4u^3}{3} - 40u^2 + 400u \right) \right] du$$

$$= \int_0^{20} \left[14u - \frac{u^3}{75} + \frac{2}{5}u^2 - 4u \right] du$$

$$= \int_0^{20} \left[10u + \frac{2}{5}u^2 - \frac{u^3}{75} \right] du$$

$$= 5u^2 + \frac{2}{15}u^3 - \frac{u^4}{300} \Big|_0^{20}$$

$$= 5(20)^2 + \frac{2}{15}(20)^3 - \frac{20^4}{300}$$

$$= (20)^2 \left[5 + \frac{2}{\cancel{15}_{3}} \times \cancel{20}^{4} - \frac{\cancel{400}}{3\cancel{00}} \right]$$

$$= 400 \left[5 + \frac{8}{3} - \frac{4}{3} \right] = 400 \left[5 + \frac{4}{3} \right]$$

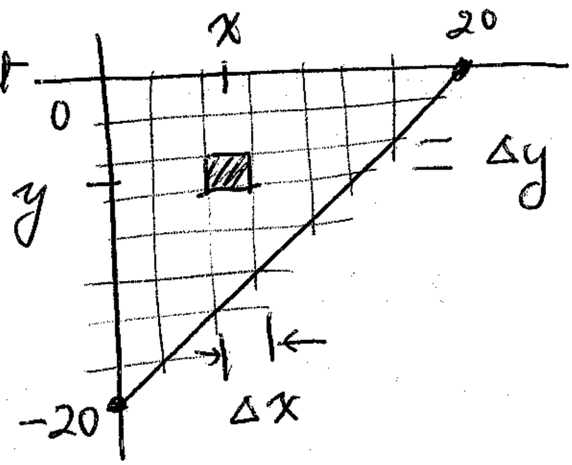
$$= 400 \left(\frac{19}{3} \right) = \frac{7600}{3} = 2533\tfrac{1}{3} \ m^3.$$

Remark: You could also do

$$V_1 = \int_0^{20} \int_{x-20}^{0} \left(14 - \frac{1}{100}(x^2+y^2) \right) dy\, dx.$$

Q2 $\rho(x,y) = x^2 + y + 20$ kg/m²

Mass of small rectangular segment at position (x,y):



$$\Delta M = \left(\begin{smallmatrix}\text{mass}\\ \text{density}\end{smallmatrix}\right) \cdot (\text{Area})$$

$$= \rho(x,y) \cdot \Delta A$$

$$= (x^2 + y + 20) \cdot \Delta A$$

Total mass

$$M \simeq \sum_{(x,y) \text{ in } F} (x^2 + y + 20)\, \Delta A$$

Set $\Delta A \to 0$ to get:

$$M = \iint_F (x^2 + y + 20)\, dA$$

Use $F = \left\{ (x,y) : \begin{matrix} 0 \le x \le 20 \\ x-20 \le y \le 0 \end{matrix} \right\}$

$$M = \int_0^{20} \int_{x-20}^{0} (x^2 + y + 20)\, dy\, dx$$

$$M = \int_0^{20} \left[x^2 y + \frac{y^2}{2} + 20y \right]_{x-20}^{0} dx$$

$$= \int_0^{20} \left(0 - \left[x^2(x-20) + \frac{(x-20)^2}{2} + 20(x-20) \right] \right) dx$$

$$= \int_0^{20} \left[-x^2(x-20) - \frac{(x-20)^2}{2} - 20(x-20) \right] dx$$

$$u = x - 20 \Rightarrow du = dx$$

$$M = \int_{-20}^{0} \left(-(u+20)^2 u - \frac{u^2}{2} - 20u \right) du$$

$$= \int_{-20}^{0} \left(-(u^2 + 40u + 400)u - \frac{1}{2}u^2 - 20u \right) du$$

$$= \int_{-20}^{0} \left(-u^3 - 40u^2 - 400u - \frac{1}{2}u^2 - 20u \right) du$$

$$= \int_{-20}^{0} \left(-u^3 - \frac{81}{2}u^2 - 420u \right) du$$

$$M = \left[-\frac{u^4}{4} - \frac{\cancel{81}^{27}}{2(\cancel{3})} u^3 - \frac{420u^2}{2} \right]_{-20}^{0}$$

$$= \frac{(-20)^4}{4} + \frac{27}{2}(-20)^3 + 210(-20)^2$$

$$= (20)^2 \left[\frac{(20)^2}{4} + \frac{27}{\cancel{2}} \cancel{(-20)}^{-10} + 210 \right]$$

$$= 400 \left[\frac{400}{4} - 270 + 210 \right]$$

$$= 400 \left[100 - 60 \right]$$

$$= 400(40) = 16000 \text{ kg}.$$

3(a) $R = \underbrace{[-2,2]}_{x} \times \underbrace{[-1,0]}_{y}$

$$\int_{-2}^{2}\int_{-1}^{0}(6x^2y + xe^y)\,dy\,dx = \int_{-2}^{2}\left[3x^2y^2 + xe^y\right]_{-1}^{0}dx$$

$$= \int_{-2}^{2}\left[0 + x - (3x^2 + xe^{-1})\right]dx$$

$$= \left[\frac{x^2}{2} - \frac{\cancel{3}x^3}{\cancel{3}} - \frac{x^2}{2}e^{-1}\right]_{-2}^{2}$$

$$= \left[\cancel{2} - 8 - \cancel{2e^{-1}} - (\cancel{2} + 8 - \cancel{2e^{-1}})\right] = -16$$

OR :

$$\int_{-1}^{0}\int_{-2}^{2}(6x^2y + xe^y)\,dx\,dy = \int_{-1}^{0}\left[2x^3y + \frac{x^2}{2}e^y\right]_{-2}^{2}dy$$

$$= \int_{-1}^{0}\left[16y + \cancel{2e^y} - (-16y + \cancel{2e^y})\right]dy$$

$$= \int_{-1}^{0}32y\,dy = \left[16y^2\right]_{-1}^{0} = -16$$

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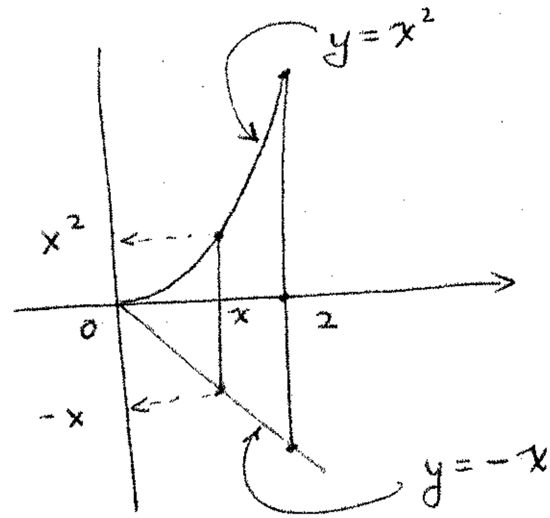
Q3(b)



$R:\quad 0 \leq x \leq 2$

$\quad\;\; -x \leq y \leq x^2$

$$\int_0^2 \int_{-x}^{x^2} (6x^2y + xe^y)\,dy\,dx$$

$$= \int_0^2 \left[3x^2y^2 + xe^y \right]_{-x}^{x^2} dx$$

$$= \int_0^2 3x^6 + xe^{x^2} - (3x^4 + xe^{-x})\,dx$$

$$= \int_0^2 (3x^6 - 3x^4)\,dx + \underbrace{\int_0^2 xe^{x^2}\,dx}_{\text{substitution}} - \underbrace{\int_0^2 xe^{-x}\,dx}_{\text{~~par~~ Integration by parts.}}$$

$w = x^2$

$$\int_0^2 3x^6 - 3x^4\,dx = \left[\frac{3x^7}{7} - \frac{3}{5}x^5 \right]_0^2$$

$$= \frac{3(2^7)}{7} - \frac{3}{5}(2)^5 = \frac{15(128) - 21(32)}{35}$$

$$= \frac{1920 - 672}{35} = \frac{1248}{35} = 35\frac{23}{35}$$

$$\int_0^2 x e^{x^2} dx = \int_0^4 e^w \cdot \tfrac{1}{2} dw = \tfrac{1}{2} e^w \Big|_0^4$$

$w = x^2$ $\qquad = \tfrac{1}{2}(e^4 - 1)$

$dw = 2x\,dx$

$$\int_0^2 \underbrace{x}_{u} \underbrace{e^{-x}\,dx}_{dv} = \left[-xe^{-x}\right]_0^2 - \int_0^2 -e^{-x}\,dx$$

$$= -2e^{-2} + \left[-e^{-x}\right]_0^2$$

$u = x$

$du = dx$ $\qquad = -2e^{-2} + \left[-e^{-2} + 1\right]$

$dv = e^{-x}\,dx$

$v = -e^{-x}$ $\qquad = 1 - 3e^{-2}$

$$\iint_R \left(6x^2y + xe^y\right) dA$$

$$= 35\tfrac{23}{35} + \left(\tfrac{1}{2}e^4 - \tfrac{1}{2}\right) - \left(1 - 3e^{-2}\right)$$

$$= 34\tfrac{11}{70} + \tfrac{1}{2}e^4 + 3e^{-2}$$